EMPIRICAL EQUATION FOR THE FUNDAMENTAL NATURAL PERIOD

by

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SYNOPSIS

An empirical method has been developed to predict the fundamental natural period of a multistory, multibay frame. Here it is first required to calculate the average values of beam stiffness, column stiffness, story height and the mass per story. Then by reading a value from the chart, which is good for all types of structure, the natural period can be determined with greatly improved accuracy over the conventional formulae. Rationale of this method is also discussed, followed by example calculations on both regular and irregular types of structure.

INTRODUCTION

The natural period in the fundamental mode is one of the most important structural parameters which will describe the dynamic characteristics of a frame. The exact value of the fundamental natural period of an analytical frame can be evaluated if the equations of motion are constructed, by employing, most conveniently, one of the iterative methods(9), with the aid of a computer.

In a practical design, however, a so called, rigorous analysis is often time consuming and an empirical formula is preferred to. For this purpose a number of such formulae have been proposed (2, 4, 7, 8, 10, 11, 12, 13, 15, 16, 17); some of them are adopted in the building codes(8, 12, 17) or recommended to use by the authorities (11). However, the estimates given by any of these empirical formulae have been rather crude, resulting in some cases in more than 100% error(4).

In this paper, an improved method to predict the fundamental natural period of a multistory, multibay frame is proposed. Then, a discussion to explain the construction of the proposed formula will be given. A consideration for irregular types of frames will also be presented. The last portion of the paper demonstrates the use of the present method and the comparison of values obtained from the present method with the true values of the natural period.

PROPOSED EMPIRICAL FORMULA FOR THE FUNDAMENTAL NATURAL PERIOD

The most practical way of setting up an analytical model of an actual frame is by its skeltons with masses concentrated at floor levels. The number of stories, N_s, and the number of bays, N_b, as well as the height of each story and the length of each bay in the

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model correspond to those of the original frame (member length will be denoted by L in the subsequent discussions). The members in the model are assumed to be massless but have the same stiffness (E: elastic modulus; and I: moment of inertia) as those of the actual frame. The masses, m_j, are concentrated at the floor level and can only sway horizontally. The base of the bottom story columns are fixed to the foundation.

An empirical equation for the calculation of the fundamental natural period, T_1 (sec), of a regular type of frame is proposed as:

$$T_1 = T_0 \, h \sqrt{\frac{\beta}{\alpha}} \cdot \frac{N_s}{10} \qquad \text{--- (1)}$$

where T_0: the value read from the chart given in Fig. 1a or 1b depending upon the value of γ, where

$$\gamma = K_b/K_c. \qquad \text{--- (2)}$$

In the above, K_b is the average stiffness of beams, i.e.,

$$K_b = \left\{ \sum_{\text{for all beams}} \left(\frac{EI}{L}\right)_b \right\} / N_s N_b \qquad \text{--- (3)}$$

where $(EI/L)_b$ is the stiffness of an individual beam; and K_c is the average stiffness of columns; i.e.,

$$K_c = \left\{ \sum_{\text{for all columns}} \left(\frac{EI}{L}\right)_c \right\} / N_s (N_b + 1) \qquad \text{--- (4)}$$

where $(EI/L)_c$ is the stiffness of an individual column,

h: the ratio of the average story height to a standard height of 12 feet = 144 inches (366 cm); i.e.

$$h = H/144N_s \qquad \text{--- (5)}$$

in which, H is the overall height (in inches) of the structure.

α: the ratio of the average stiffness of columns, K_c to a standard value of stiffness of 0.500×10^6 kip-in ($.575 \times 10^6$ ton cm); i.e.

$$\alpha = 2K_c/10^6 , \qquad \text{--- (6)}$$

and

β: the ratio of the average mass per story per column, m, to a standard value of mass of 70/g kip-sec^2/in (31.7/g ton sec^2/cm) adjusted by the number of bays; i.e.

$$\beta = \frac{mg}{70} \cdot \frac{N_b + 0.4}{N_b} \qquad \text{--- (7)}$$

in which, g is the acceleration of gravity ($\simeq$ 386 in/sec^2 or 980 cm/sec^2).

EXPLANATION OF THE PRESENT FORMULA

The present method to predict the fundamental natural period is an empirical method, and it is not possible to verify its validity in a rigorous manner. However, the following explanation may help to understand the construction of the present formula.

The governing equation of the natural period, T, of the system described above is written in the following manner.

$$[G]\{x\} = \left(\frac{2\pi}{T}\right)^2 [M]\{x\} \qquad \text{--- (8)}$$

where [G] is the frame stiffness matrix, [M] is the mass matrix and {x} is the floor level lateral translation vector.

The values of T_0 in Fig. 1a or 1b were obtained as the fundamental natural periods of a series of standard 10-story, 2-bay frames by solving Eq. 8 in a rigorous manner. In these standard frames, the mass distribution (average mass per story per column is 58.3/g kip sec^2/in, or 26.4/g ton sec^2/cm), the distribution of column stiffnesses (average value of EI/L is .500 x 10^6 kip in, or .575 x 10^6 ton cm), and the story heights (average height is 144 in., or 366 cm) were kept constant and only the beam stiffnesses were varied. Thus the chart is made for T_0 against γ where γ is the ratio of the average beam stiffness to the average column stiffness as defined in Eq. 2.

The effect of the distribution of column stiffnesses on the fundamental natural period was found minor. That is, if other variables are kept constant, the fundamental natural period of a frame is almost the same regardless of the stiffness distribution of columns (within a practical range) as far as its average stiffness is kept the same. Similarly the effect of the distribution of beam stiffnesses on the fundamental natural period was minor, so as the distribution of masses as far as their average values were kept constant. Therefore, these factors are ignored in the present equation.

An element of the structural stiffness matrix [G] has a value expressed in a form, $\Sigma\, p(EI/L^3)_c$. Here, p is a nondimensionalized quantity and is a function of $(EI/L)_b/(EI/L)_c$, where $(EI/L)_b$ and $(EI/L)_c$ are stiffnesses of appropriate beam and column members, respectively; and $(EI/L^3)_c$ are the quantities, EI/L^3, with respect to column members. Thus if the quantities of EI/L of all column and beam members are increased or decreased proportionally to the original values with almost the same ratio, say α times the original values on the average, the new stiffness matrix would approximately be expressed as a[G], where, since the quantities p would remain the same,

$$a = \frac{\text{Average value of } (EI/L^3)_c \text{ of the new frame}}{\text{Average value of } (EI/L^3)_c \text{ of the old frame}}$$

$$\simeq \alpha \left(\frac{\text{Average story height of the old frame}}{\text{Average story height of the new frame}}\right)^2 \qquad \text{--- (9)}$$

The equations describing the natural period of the new frame can now be written as:

$$a\,[G]\{x\} = \left(\frac{2\pi}{T}\right)^2 [M]\{x\} \qquad \text{--- (10)}$$

This indicates that the natural period of the new frame, T_1, is $1/\sqrt{a}$ times that of the original frame, T_0. If T_0 is assumed to be the value obtained from the chart given in Fig. 1a or 1b entering γ which is calculated on the new frame, the quantity a in Eq. 9 is now expressed as :

$$a \simeq \alpha / h^2 \qquad \text{--- (11)}$$

where α and h as defined in Eq. 6 and 5, respectively. Thus,

$$T_1 = T_0/\sqrt{a} = T_0 h/\sqrt{\alpha} \qquad \text{--- (12)}$$

On the other hand, if all masses are increased or decreased proportionally to the original values with approximately the same ratio, say b times the original value on the average, the equations for the natural period of the new frame can be written as

$$[G]\ \{x\} \simeq \left(\frac{2\pi}{T}\right)^2\ b\ [M]\ \{x\} \qquad \text{--- (13)}$$

Thus the natural period of the new frame, T_1, will be approximately $\sqrt{b}$ times that of the original frame, T_0. If T_0 is obtained from the chart in Fig. 1a or 1b, b may be calculated as:

$$b = \frac{\text{Average mass per story of the new frame}}{\text{Average mass per story of the old frame}} = \frac{mg}{58.3} \qquad \text{---(14)}$$

where m is the average mass per story per column of the new frame. Thus,

$$T_1 \simeq T_0 \sqrt{b} \qquad \text{--- (15)}$$

Further it was observed that even if the values of the mass per story per column are the same in two frames of the different numbers of bays, the less the number of bays, the longer the natural period. It was found that the natural period is almost proportional to $\sqrt{(N_b + 0.4)/N_b}$ when other parameters are kept constant. Combining the preceding two effects, new quantity, β, was defined as:

$$\beta = \frac{mg}{70} \cdot \frac{N_b + 0.4}{N_b} \qquad \text{--- (16)}$$

This definition was selected so that β becomes equal to 1.0 when $m = \frac{58.3}{g}$ kip sec^2/in and $N_b = 2$ as in the standard frames. When the definition of Eq. 16 is used, the standard value of mass per story per column may be regarded as 70/g kip sec^2/in. Eq. 15 is now replaced by the next equation.

$$T_1 = T_0 \sqrt{\beta} \qquad \text{--- (17)}$$

If, both the structural stiffnesses and mass distribution are varied, the fundamental natural period of the new frame (of any number of bays) will be obtained, by superposing Eqs. 12 and 17 as:

$$T_1 = T_0 h \sqrt{\frac{\beta}{\alpha}} \qquad \text{--- (18)}$$

Finally, the effect of number of stories was investigated. The observation indicated that the change in the fundamental natural period is approximately proportional to the number of stories 'if' the quantities, γ, α, β and h are kept constant. Thus the final form of the equation to predict the fundamental natural period was proposed as:

$$T_1 = T_0 h \sqrt{\frac{\beta}{\alpha}} \cdot \frac{N_s}{10}$$

which is the formula shown in Eq. 1.

CONSIDERATION OF IRREGULAR FRAMES

Eq. 1 was proposed for the calculation of the fundamental natural period of the regular type of frames. If the frame contains shearwalls and/or non-rigidly framed members, the quantities defined above must be modified if a better prediction is desired.

(1) Frame Containing Shearwalls

Such a frame may be represented by the model shown in Fig. 2. Rigid stubs simulate the wall width effect(6). In this case the stiffness of the beam members attached to the shearwall is calculated using the effective length, L_e, instead of the column center to center length, L, as used in regular type of frames (Eq. 3). Considering the fact that because of the forced swayed position, the restraining moments produced by such beams are greater than those produced by the ordinary beams of the same length, the effective length, L_e, was defined as:

$$L_e = \{1 - 2(\lambda_1 + \lambda_2)\} L \qquad \text{--- (19)}$$

where $\lambda_1 L$ and $\lambda_2 L$ represent the lengths of the rigid stubs at the left and right ends of the beam, respectively. The fundamental natural period may then be determined using Eq. 1.

(2) Frame Containing Pinned End Members

If a beam is connected to its supporting columns by pinned joints, the bending stiffness of this beam is taken as zero. If only one end has a pinned connection, the stiffness of the beam may be calculated using an effective length, L_e, equal to twice the actual length.

If interior columns have pinned ends as shown in Fig. 3a, these columns may completely be ignored. Therefore, the example in Fig. 3a may be regarded as a single bay frame ($N_b = 1$ is assumed in Eqs. 3 and 4). The beam stiffness based on the average moment of inertia of the left and right beams with the length of beam equal to the total of the two beam lengths.

If exterior columns have pinned ends as shown in Fig. 3b, the example is again considered as a single bay frame ($N_b = 1$ in Eq. 4). The average beam stiffness, K_b, is calculated as:

$$K_b = \Big[\sum_{\substack{\text{for all beams}\\ \text{in regular bays}}} \left(\frac{EI}{L}\right)_b + \sum_{\substack{\text{for all beams}\\ \text{in special bays}}} \left(\frac{EI}{L_e}\right) \Big] / N_s N_b \qquad \text{--- (20)}$$

in which a special bay is the exterior bay where the exterior ends of the beams are pinned. Regular bays are all other bays. The effective beam lengths, L_e, in special bays are taken as twice the actual beam lengths. In general, N_b is the number of bays which is equal to the actual number of bays minus the number of special bays.

EXAMPLES

(1) Example Calculation on Frame #1

Dimension and structural properties of Frame #1 are shown in Table 1 and in Fig. 4. This frame may be regarded as a regular type of frame. The calculations proceed as follows:

$$K_b = \Sigma \left(\frac{EI}{L}\right)_b / N_s N_b = .197 \times 10^6 \text{ kip in.}$$

$$K_c = \Sigma \left(\frac{EI}{L}\right)_c / N_s (N_b + 1) = .527 \times 10^6 \text{ kip in.}$$

Thus, $\gamma = K_b/K_c = .375$

Then, T_0 is read from the chart in Fig. 1a as: $T_0 = 1.83$ sec.
The average story height is 144", thus $h = 144/144 = 1.0$.
The quantity, α, is given by: $\alpha = 2K_c \times 10^{-6} = 1.053$
The average weight per story per column is 95 kips, and $N_b = 4$, then β is calculated as:

$$\beta = \frac{95}{70} \times \frac{4 + 0.4}{4} = 1.49$$

Substituting above values into Eq. 1, the fundamental natural period of Frame #1 is calculated as:

$$T_1 = T_0 h \sqrt{\frac{\beta}{\alpha}} \cdot \frac{N_s}{10} = 1.83 \times 1.0 \times \sqrt{\frac{1.49}{1.053}} \times \frac{10}{10} = 2.18 \text{ sec.}$$

The rigorous calculation resulted in $T_1 = 2.25$ sec., which indicates the error in the empirical formula is about 3% in this case.

(2) Example Calculation on Frame #2

Frame #2 contains a shearwall as well as exterior columns whose ends are pin connected. Dimensions and necessary properties are shown in Fig. 5 and Table 2. The empirical formula is used to estimate the fundamental natural period in the following manner.

$$K_b = \left[\sum_{\text{left bay}} \left(\frac{EI}{L_e}\right)_b + \sum_{\text{right bay}} \left(\frac{EI}{L_e}\right) \right] / N_s N_b$$

$$= (2.65 + 1.33) \times 10^6/10 \times 1 = .398 \times 10^6 \text{ kip in.}$$

In the above, the effective length, L_e, for the beams in the left hand bay is taken, according to Eq. 19, as: $L_e = 360 - 2 \times 60 = 240$ inches, and for the beams in the right hand bay: $L_e = 2 \times (360 - 2 \times 60) = 480$ inches. Now, $K_c = \Sigma \left(\frac{EI}{L}\right)_c / N_s(N_b + 1) = 4.736 \times 10^6$ kip in.

Thus, $$\gamma = \frac{K_b}{K_c} = \frac{.398}{4.736} = .084$$

Then, T_0 is read from the chart in Fig. 1b as: $T_0 = 3.34$ sec.
The quantiy, α, is given by: $\alpha = 2K_c \times 10^{-6} = 9.47$

The average weight per story per column is 87 kips and $N_b = 1$, therefore:

$$\beta = \frac{87}{70} \cdot \frac{1 + 0.4}{1} = 1.74$$

The average story height is 144", thus: h = 1.0
Substituting these values into Eq. 1, the fundamental natural period of Frame #2 is calculated as:

$$T_1 = T_0 h \sqrt{\frac{\beta}{\alpha}} \cdot \frac{N_s}{10}$$

$$= 3.34 \times 1.0 \sqrt{\frac{1.74}{9.47}} \times \frac{10}{10} = 1.43 \text{ sec.}$$

The rigorous calculation indicated a natural period of 1.54 sec. The error in the empirical formula when applied to this rather irregular type of frame is approximately 7%.

ACCURACY OF THE PRESENT METHOD

The proposed formula, Eq. 1, has been tested on many regular type of frames as well as some irregular type of frames. Some of them are shown in Table 3, as well as in Fig. 6. Example frames listed here include 5 frames used by Goel(3), 4 frames by Blume(1), 1 frame by Lionberger and Weaver(5), 1 frame by Newmark and Rosenblueth(9), and 11 frames by Suko and Adams(14). The frames tested ranged from 2 to 40 stories and 1 to 4 bays. The deviation from the rigorously calculated value is found to be less than 5% in most cases.

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TABLE 1. STRUCTURAL PROPERTIES OF FRAME #1

Story or Floor	Weight per Floor kips	(tons)	Moment of Inertia, $inch^4$ ($centimeter^4$) Column[a]		Beam[b]	
1	250	(113)	1200	(49800)	2000	(83100)
2	500	(226)	1200	(49800)	2000	(83100)
3	500	(226)	1200	(49800)	2000	(83100)
4	500	(226)	2200	(91500)	2000	(83100)
5	500	(226)	2200	(91500)	2000	(83100)
6	500	(226)	2200	(91500)	2000	(83100)
7	500	(226)	3300	(137000)	2000	(83100)
8	500	(226)	3300	(137000)	2000	(83100)
9	500	(226)	4400	(183000)	2000	(83100)
10	500	(226)	4400	(183000)	2000	(83100)

a) Same for all columns within a story.
b) Same for all beams within a floor
E (Modulus of Elacity) taken as 29600 kip/in^2 (2080 ton/cm^2)

TABLE 2. STRUCTURAL PROPERTIES OF FRAME #2

Story or Floor	Weight per Floor kips	(tons)	EI (Modulus of Elasticity x Moment of Inertia) kip in^2 (ton cm^2) Column		Shearwall		Beam[a]	
1	120	(54.2)	71	(208)	568	(1660)	44	(129)
2	180	(81.4)	71	(208)	568	(1660)	44	(129)
3	180	(81.4)	71	(208)	568	(1660)	44	(129)
4	180	(81.4)	130	(380)	1042	(3045)	59	(173)
5	180	(81.4)	130	(380)	1042	(3045)	59	(173)
6	180	(81.4)	130	(380)	1042	(3045)	59	(173)
7	180	(81.4)	195	(570)	1563	(4560)	74	(216)
8	180	(81.4)	195	(570)	1563	(4560)	74	(216)
9	180	(81.4)	261	(761)	2084	(6095)	89	(259)
10	180	(81.4)	261	(761)	2084	(6095)	89	(259)

a) Same for the left and right beams within a floor.

TABLE 3. COMPARISON OF THE NATURAL PERIODS BY THE PRESENT METHOD WITH THOSE BY THE RIGOROUS CALCULATION

Frame Used By	Size (Story x Bay)	Rigorous Calculation (Sec.)	Present Method (Sec.)	Deviation
Blume	5 x 1	1.53	1.44	.059
"	10 x 1	2.57	2.51	.023
"	15 x 1	3.43	3.35	.023
"	20 x 1	4.14	4.17	.007
Goel (taper model)	10 x 1	1.25	1.25	.000
" (taper model)	25 x 1	2.27	2.29	.009
" (taper model)	40 x 1	3.00	3.04	.013
" (uniform model)	10 x 1	1.25	1.25	.000
" (uniform model)	25 x 1	2.27	2.32	.022
Lionberger & Weaver	10 x 2	2.18	2.18	.000
Newmark & Rosenblueth	2 x 1	0.31	0.29	.065
Suko & Adams	5 x 3	0.54	0.53	.019
"	5 x 3	0.63	0.61	.032
"	5 x 3	0.78	0.76	.026
"	5 x 3	0.98	0.94	.041
"	5 x 4	1.17	1.15	.017
"	10 x 2	1.16	1.18	.017
"	10 x 2	1.48	1.44	.027
"	10 x 4	2.25	2.20	.022
" (with a shearwall)	10 x 2	1.60	1.55	.031
" (with a shearwall)	10 x 2	2.57	2.53	.016
" (irregular type)	10 x 2	1.54	1.43	.071

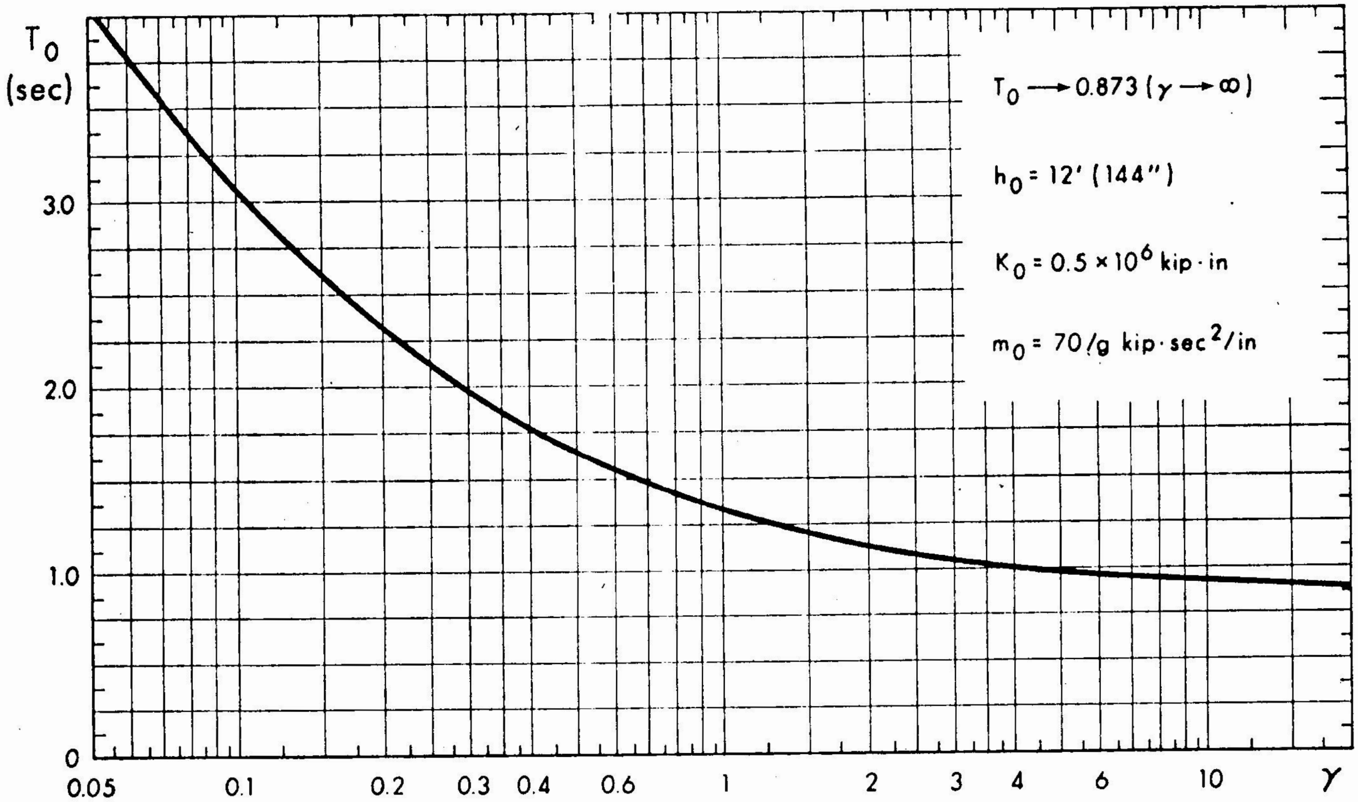

Fig. 1a Value of T_0 Used in Eq. 1

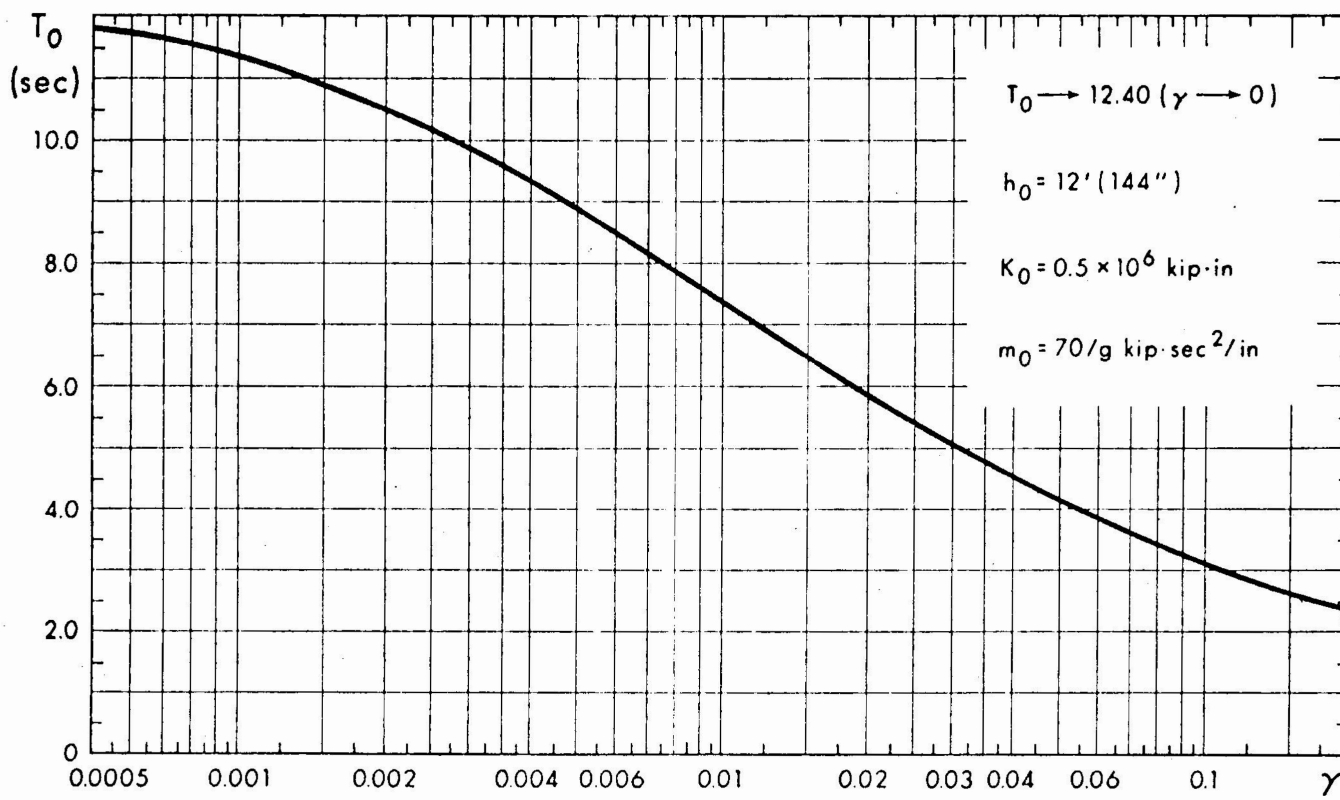

Fig. 1b Value of T_0 Used in Eq. 1

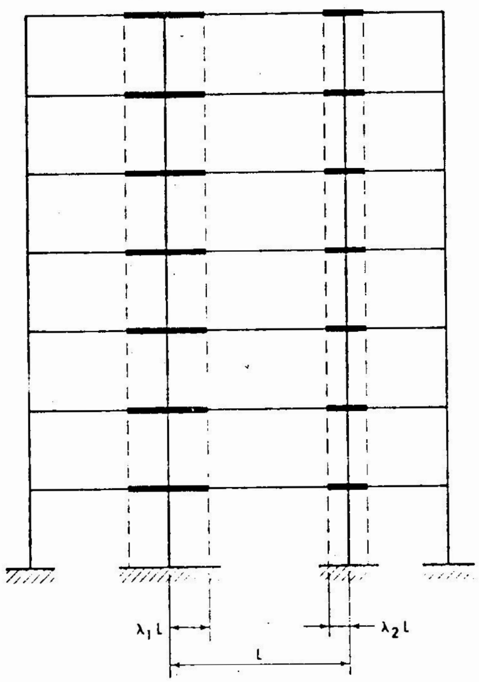

Fig. 2

Frame with Shearwalls

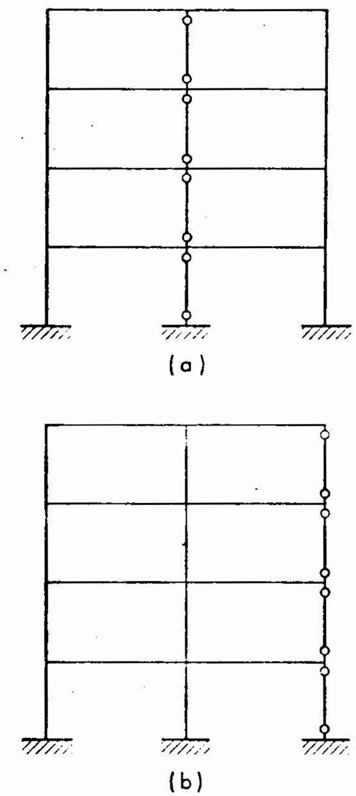

Fig. 3

Frames with Pinned-End Columns

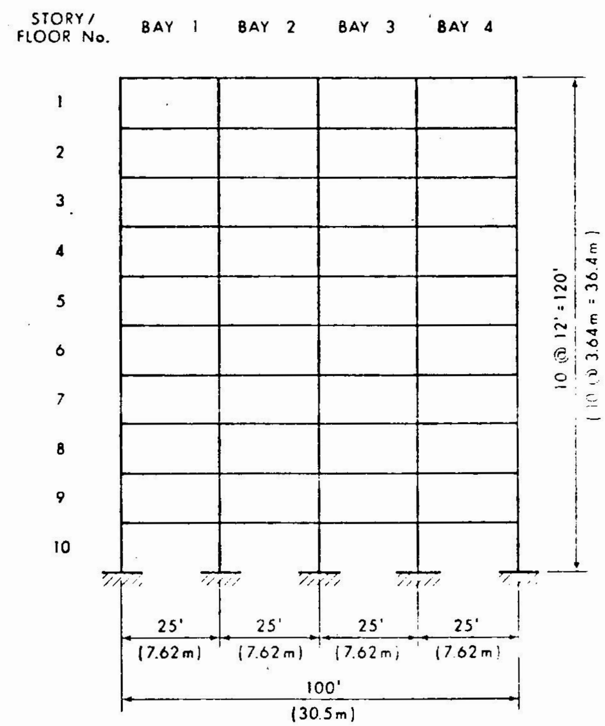

Fig. 4 Frame #1

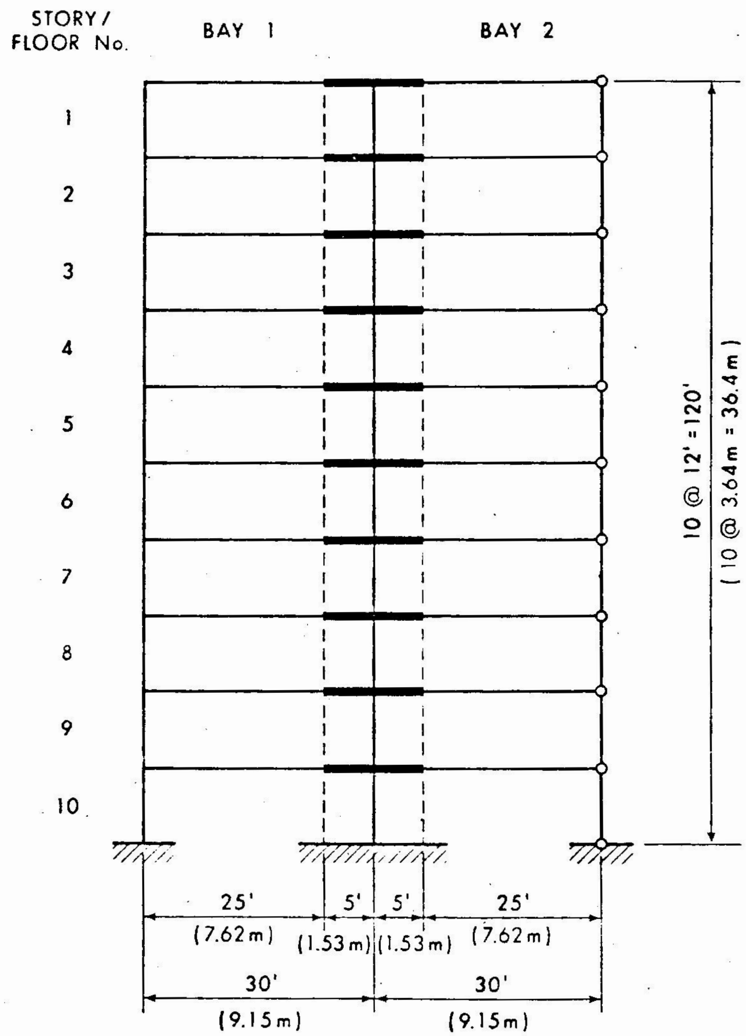

Fig. 5 Frame #2

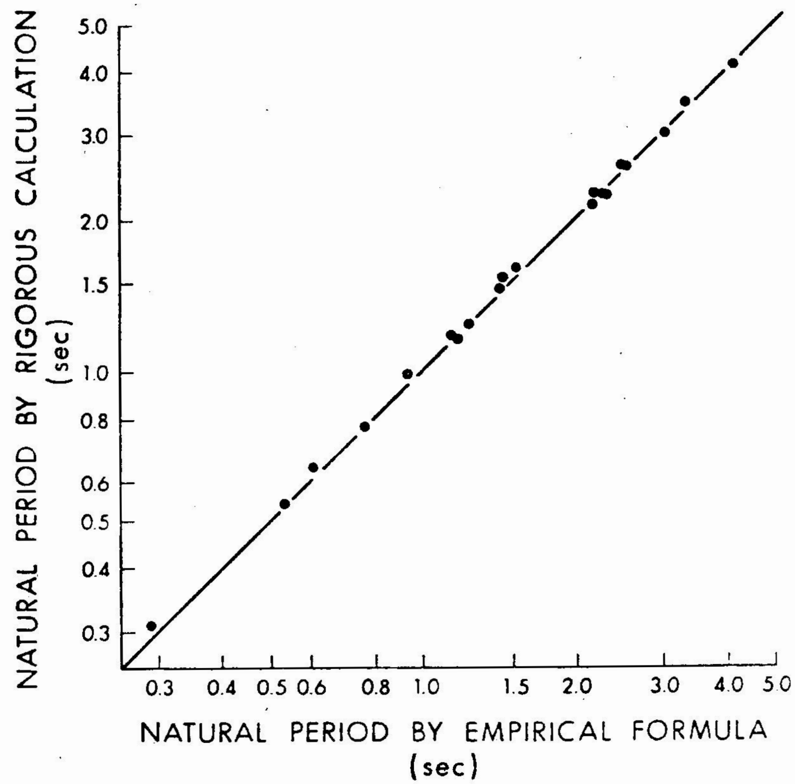

Fig. 6 Correlation Between Natural Period by Rigorous Calculation and Natural Period by Empirical Formula